A STUDY OF RELATIVE MOTION IN CONNECTION WITH CLASSICAL MECHANICS

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H. ZANSTRA

A THESIS

SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL OF THE UNIVERSITY OF MINNESOTA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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Abstract

Derivation of equations of motion of the second order, containing relative coordinates only.—This article is a continuation of the work of many authors, among others Berkeley, Neumann, Maxwell, Lange, Mach, Boltzmann, Föppl, and Kolkmeyer, whose work is briefly discussed. The main difficulty in Newton's mechanics seems to be to define the system of axes (inertial system) to which the motion is referred. Three solutions have been proposed. Föppl defined the system on a relative basis. Another and apparently more direct solution is to refer the motion of particles to other particles, that is, to use "relative coordinates" in the equations of motion. A third solution would be to obtain invariant equations, the same for all systems of reference. Introducing an hypothesis stated by Föppl, that for a system with its origin at the center of mass the angular momentum of the universe vanishes, invariant equations are obtained from the Newtonian, (though this is sometimes denied to be possible), and they degenerate into the Newtonian form if certain functions of coordinates and velocities occurring in them become zero by a suitable choice of the system of reference. It is believed, however, that such equations do not offer a complete solution of the problem of relativation of motion so as to satisfy the physicist; whereas the introduction of relative coordinates does. In this article this second solution has been applied to the Newtonian theory. Equations of motion are derived by introducing Föppl's hypothesis, which are of the second order and contain relative coordinates only. These equations may be taken as the basis for a complete system of mechanics. As examples the systems of Ptolemy and of Kepler are worked out.

Application of relative coordinates to Einstein's theory of gravitation.—It is believed that the possibility of introducing relative coordinates is not limited to the Newtonian theory. As an example of the application of these coordinates to Einstein's theories, the case of the motion of a heavy and a light particle is considered and it is suggested that an hypothesis similar to Föppl's may give equations of motion into which only the distance apart of the particles measured by an observer on the heavier, enters.

1. *The use of relative coordinates to describe the observed motion.* It is well known that Newton introduced the idea of absolute motion in the system of mechanics he created. At the same time, however, he stated explicitly that the motion of bodies in space, such as we observe, is a relative one. The motion of bodies is given by the position of their parts at different instants, and the observed motion is relative because the observed position is relative. Let us inquire as to the sense in which this is to be understood. We introduce for this purpose the customary idea of a material point, that is of a body or part of a body of small dimensions, which, as far as its position is concerned, may be treated

as a mathematical point. If there is a system of n points ($n>2$), to which our observation is confined, the complete specification of the distances of all these points requires the determination of only $3n-6$ of them, suitably chosen. We might e.g. determine for any three points 1, 2 and 3 the sides of the triangle they form, and for any subsequent point ν the three distances of that point from the corners of that triangle. Some of the distances might be replaced by measured angles between lines joining the points, or between the planes of two adjacent triangles. Our conclusion is then, that the relative position of the n material points (or as is sometimes said: the configuration of the point system) is given by $3n-6$ independent coordinates, to be determined by observation, the term "coordinates" being used in the generalised sense. In every practical case of observation the position of material points is relative in the sense understood here.

Coordinates serving to fix the relative position of material points will be designated as relative coordinates. We aim to solve the difficulties connected with the idea of absolute motion in Newton's mechanics by introducing these relative coordinates in the equations of motion.

2. *An equation of motion in relative coordinates, obtained from the differential equations of Newton's mechanics.* The relative coordinates we are about to introduce in this paragraph are referred to a rectangular system of axes XYZ. This system XYZ is defined in such a way that its origin permanently coincides with any one of the material points, which will be designated as point 1; further we shall pass its axis of X permanently through any second point 2, and the XY plane permanently through any third point 3. There is ambiguity as regards the positive direction of the axis of Y, but if once chosen it remains definite throughout.

We have by the definition of our axes

$$x_1=0, \quad y_1=0, \quad z_1=0, \quad y_2=0, \quad z_2=0, \quad z_3=0. \tag{1}$$

The $3n-6$ independent coordinates, specifying the relative position of n material points, are: x_2, x_3, y_3; and x_ν, y_ν, z_ν for $\nu>3$.

We shall start from the equations of motion of Newton's mechanics:

$$m_\nu \ddot{x}_\nu' = X_\nu', \quad m_\nu \ddot{y}_\nu' = Y_\nu', \quad m_\nu \ddot{z}_\nu' = Z_\nu'. \tag{2}$$

In these equations the force components X_ν', Y_ν' and Z_ν' represent definite functions of the relative coordinates of all the particles (like the force of gravitation), though occasionally they might be definite functions of the time, or the first time derivative of the relative coordinates might occur in them. In our discussions we shall not use the term force in any other sense.

We assume these Eqs. (2) to hold in a system of axes $X'Y'Z'$. We proceed to transform them to the relative system of coordinates XYZ. As we are perfectly free in choosing the direction of the original axes $X'Y'Z'$, we shall take them *momentarily* parallel to the new axes XYZ. The Eqs. (2) are then, transformed:

$$\left.\begin{aligned}
&\ddot{x}_\nu-(\dot{q}z_\nu-\dot{r}y_\nu)-(q^2+r^2)x_\nu+p(qy_\nu+rx_\nu)-2(q\dot{z}_\nu-r\dot{y}_\nu)-a=X_\nu/m_\nu;\\
&\ddot{y}_\nu-(\dot{r}x_\nu-\dot{p}z_\nu)-(r^2+p^2)y_\nu+q(rz_\nu+px_\nu)-2(r\dot{x}_\nu-p\dot{z}_\nu)-b=Y_\nu/m_\nu;\\
&\ddot{z}_\nu-(\dot{p}y_\nu-\dot{q}z_\nu)-(p^2+q^2)z_\nu+r(pz_\nu+qy_\nu)-2(p\dot{y}_\nu-q\dot{x}_\nu)-c=Z_\nu/m_\nu;
\end{aligned}\right\}\quad(3)$$

where a, b, c are the components of the acceleration of the origin of the old system and p, q, r the components of its angular velocity, all with respect to XYZ.

If these Eqs. (3) did not contain the six quantities a, b, c, p, q, r, there would occur in them only relative coordinates. As it stands, however, our problem is not solved, unless we express those quantities in terms of the relative coordinates. But this can readily be done by substituting the six conditions (1) in (3). Carrying out this substitution, we obtain:

$$\left.\begin{aligned}
&\ddot{x}_\nu-(\dot{q}z_\nu-\dot{r}y_\nu)-(q^2+r^2)x_\nu+p(qy_\nu+rz_\nu)-2(q\dot{r}_\nu-r\dot{y}_\nu)-a=X_\nu/m_\nu,\\
&\text{with 2 similar expressions for } y- \text{ and } z-\text{axis.}\quad(\nu=4\ldots n)\\
&\ddot{x}_2-(q^2+r^2)x_2-a=X_2/m_2,\\
&\ddot{x}_3+\dot{r}y_3-(q^2+r^2)x_3+pqy_3+2r\dot{y}_3-a=X_2/m_3,\\
&\ddot{y}_3-\dot{r}x_3-(r^2+p^2)y_3+qpx_3-2r\dot{x}_3-b=Y_3/m_3.
\end{aligned}\right\}\quad(4a)$$

$$\left.\begin{aligned}
&a=-X_1/m_1,\ b=-Y_1/m_1,\ c=-Z_1/m_1,\\
&-\dot{r}x_2+qpx_2-2r\dot{x}_2-b=Y_2/m_2,\ \dot{q}x_2+rpx_2+2q\dot{x}_2-c=Z_2/m_2,\\
&-(\dot{p}y_3-\dot{q}x_3)+r(px_3+qy_3)-2(p\dot{y}_3-q\dot{x}_3)-c=Z_3/m_3.
\end{aligned}\right\}\quad(4b)$$

The substitution has not altered the Eqs. (3) for $\nu>3$. The substitution for $\nu=1$, 2 and 3 gave the last three equations (4a) and the equations (4b).

Recalling that X_ν, Y_ν and Z_ν stand for functions of the relative coordinates, we may look upon the Eqs. (4a) as being $3n-6$ independent relations among the $3n-6$ independent coordinates, mentioned above: x_2, x_3, y_3, x , y_ν, z_ν $(\nu>3)$, containing, moreover, 6 auxiliary quantities a, b, c, p, q, r. But these 6 auxiliary quantities are expressed in terms of the coordinates by means of the differential Eqs. (4b), and when these six quantities are eliminated from (4a) and (4b), we are left with $3n-6$ *independent differential equations for the* $3n-6$ *independent coordinates*, therefore our problem is theoretically solved.

The system of differential equations (4) is of order $6n-9$. [See §8 (a)]. Therefore it can be reduced to *$3n-9$ equations of the second order and 3 equations of the third order* in which only the $3n-6$ independent coordinates occur. For our purpose it is not necessary to make this

reduction. We shall therefore leave equations (4a) and (4b) as they stand. However, we should keep in mind that our point of view forbids that we interpret the six quantities a, b, c, p, q and r as components of absolute acceleration and angular velocity. They are to be regarded merely as auxiliary quantities which can be eliminated.

To describe the observed motion in a system of material points, A. Föppl[1] has suggested the system of axes XYZ as used here, though he did not introduce the relative coordinates in the equations of motion. On the other hand N. H. Kolkmeyer[2] suggested and worked out the idea of eliminating quantities like a, b, c, p, q and r by expressing them in terms of coordinates.

The combination of these two ideas leads to the equations (4a) and (4b), which by their rather simple form and their general nature are suitable for a general discussion.

Kolkmeyer's purpose is (among others) to introduce relativity of motion in Newton's mechanics without an additional assumption. Therefore his work may lead to much the same conclusions as the equations of this paragraph. In the work of the following paragraphs, however, we shall deviate from the basis on which Kolkmeyer has built, by introducing an hypothesis stated by Föppl.

3. *Föppl's hypothesis concerning the world system and its angular momentum.*[1] An *inertial system*[3] will be defined as a system of axes for which the differential equations of Newton's mechanics hold:

$$m_\nu \ddot{x}_\nu' = X_\nu', \quad m_\nu \ddot{y}_\nu' = Y_\nu', \quad m_\nu \ddot{z}_\nu' = Z_\nu'. \quad (\nu = 1 \ldots n) \tag{2}$$

We assume that there are such inertial systems $X'Y'Z'$. Moreover, we assume with Föppl that the totality of matter in the universe consists of a finite number n of points, so that the undex ν includes all material points in space. Furthermore we suppose that the quantities X', Y', Z' on the righthand side obey the *law of action and reaction* for forces and couples:

$$\left.\begin{aligned} \Sigma X' = 0, \ \Sigma Y' = 0, \ \Sigma Z' = 0, \\ \Sigma(y'Z' - z'Y') = 0, \ \Sigma(z'X' - x'Z') = 0, \ \Sigma(x'y' - y'X') = 0, \end{aligned}\right\} \tag{5}$$

where Σ includes all n points.

Combining this with (2) and integrating we obtain the equations for conservation of momentum and angular momentum:

$$\left.\begin{aligned} \Sigma m\dot{x}' = T_1, \ \Sigma m\dot{y}' = T_2, \ \Sigma m\dot{z}' = T_3, \\ \Sigma m(y'\dot{z}' - z'\dot{y}') = R_1, \ \Sigma m(z'\dot{x}' - x'\dot{z}') = R_2, \ \Sigma m(x'\dot{y}' - y'\dot{x}') = R_3. \end{aligned}\right\} \tag{6}$$

[1] A. Föppl, Vorlesungen über technische Mechanik, VI, Erster Abschnitt, Die relative Bewegung.

[2] N. H. Kolkmeyer. Eliminatie van de begrippen assenstelsel, lengte en tijd uit de vergelijkingen voor de planetenbewegingen. Thesis Amsterdam 1915.

[3] This term was introduced by Lange. See §9.

The inertial system $X'Y'Z'$ can be chosen in such a way that its origin coincides permanently with the center of mass of the total system, as this restriction does not violate Eqs. (2). Such an inertial system with origin permanently at the center of mass is defined as a *principal system of reference* (Hauptbezugssystem):

$$\Sigma mx' = 0,\ \Sigma my' = 0,\ \Sigma mz' = 0. \tag{7}$$

From (7) it follows that in such a principal system $X'Y'Z'$: $T_1 = \dot{T}_2 = T_3 = 0$. In ordinary mechanics the other three constants then depend upon the distribution of velocities in the universe and are not, in general, zero, unless a special hypothesis about this velocity distribution is made.

This *hypothesis* has been made by *Föppl*: *For a principal system of reference the angular momentum of the universe vanishes, that is:*

$$\Sigma m(y'\dot{z}' - z'\dot{y}') = 0,\ \Sigma m(z'\dot{x}' - x'\dot{z}') = 0,\ \Sigma m(x'\dot{y}' - y'\dot{x}') = 0. \tag{8}$$

Föppl's hypothesis was originated for the purpose of defining inertial systems without making use of the idea of absolute motion.[4] In this way he gave a solution of the difficulties arising from this idea in Newton's mechanics. Another way of solving these difficulties is by introducing relative coordinates in the equation of motion; and Föppl's hypothesis, though not necessary for a solution (see §2), may be of great help in simplifying this problem.

We wish to illustrate this by the following problem:

To relativate the motion if the total system consists of two material points of masses m_1 and m_2, exerting on each other a force which is a function $f(r)$ of their distance r and directed along the line joining them.

We take the inertial system $X'Y'Z'$, to which the motion in ordinary mechanics is referred, with its origin permanently at the center of gravity G of the point system. Then the velocity vectors of the two particles have opposite directions.

Föppl's hypothesis states that in such a principal system of reference the total angular momentum vanishes which is only possible if the velocity vectors of both points are directed along the line joining m_1 and m_2. The motion can therefore be described as taking place along the axis of X. The equations of motion become then

$$m_1\ddot{x}_1' = f(r), \text{ and } m_2\ddot{x}_2' = -f(r).$$

Since, moreover, $x_1' - x_2' = r$, and therefore $\ddot{x}_1' - \ddot{x}_2' = \ddot{r}$, we have

$$m_{12}\ddot{r} = f(r), \tag{I}$$

where $m_{12} = m_1m_2/(m_1+m_2)$. This is the equation of motion, containing the only relative coordinate r, which fixes the relative position of the points (§1).

[4] See §6.

If Föppl's hypothesis is not introduced, we obtain instead of (I) the more complicated expression:

$$m_{12}\ddot{r}=f(r)+R^2/m_{12}r^3 \qquad \text{(II)}$$

where R is the magnitude of the angular momentum of the system with respect to G. This constant can either be expressed in terms of r and $\ddot{r}$ at one instant or be removed by differentiation. In the latter case the equation of motion becomes

$$m_{12}(\dddot{r}r+3\ddot{r}\dot{r})+[3f(r)+rf'(r)]\dot{r}=0, \qquad \text{(III)}$$

which is of the *third* order, as might be expected according to §2. On the other hand, Föppl's hypothesis enabled us to derive the Eq. (I) which is of the *second* order.

4. *The equations of motion in relative coordinates, modified by introducing Föppl's hypothesis.* We shall proceed to modify the equations of motion (4a) and (4b) of the general case, by introducing in the derivations of §2 *Föppl's* hypothesis as an additional assumption. For the sake of convenience we shall take the inertial system $X'Y'Z'$, from which we started in §2, as a principal system (see §3):

$$\Sigma mx'=0,\ \Sigma my'=0,\ \Sigma mz'=0. \qquad (7)$$

Föppl's hypothesis is then expressed by

$$\Sigma m(y'\dot{z}'-z'\dot{y}')=0,\ \Sigma m(z'\dot{x}'-x'\dot{z}')=0,\ \Sigma m(x'\dot{y}'-y'\dot{x}')=0. \qquad (8)$$

In the derivation we carried out in §2 p, q and r represent the components of the angular velocity of the original system with reference to the new system XYZ. From (7) it follows that its origin has the coordinates $\Sigma mx/\Sigma m$, $\Sigma my/\Sigma m$, $\Sigma mz/\Sigma m$. In view of this the additional condition (8) becomes

$$p[\Sigma m\Sigma m(y^2+z^2)-(\Sigma my)^2-(\Sigma mz)^2]-q(\Sigma m\Sigma mxy-\Sigma mx\Sigma my) \qquad (9)$$
$$-r(\Sigma m\Sigma mzx-\Sigma nz\Sigma mx)=\Sigma m\Sigma m(y\dot{r}-z\dot{y})-(\Sigma my\Sigma m\dot{z}-\Sigma mz\Sigma m\dot{y}),$$

with two similar expressions for the y- and z-axis.

These equations give us p, q and r as functions of the relative coordinates and their first derivatives. These functions may be substituted in the equations (4a) and (4b) of §2 and then the quantities $\dot{p}$, $\dot{q}$, $\dot{r}$ and a, b, c eliminated. We shall not carry out this process (apart from the

rather trivial elimination of a, b and c), but put the result in a form such as to indicate the elimination. The result is:

$$\left.\begin{aligned}
&\ddot{x}_\nu-(\beta z_\nu-\gamma y_\nu)-(q^2+r^2)x_\nu+p(qy_\nu+rz_\nu)-2(q\dot{z}_\nu-r\dot{y}_\nu)=\frac{X_\nu}{m_\nu}-\frac{X_1}{m_1},\\
&\quad\text{with two similar expressions for } y\text{- and } z\text{-axis}\quad(\nu=4\ldots n)\\
&\ddot{x}_2-(q^2+r^2)x_2=\frac{X_2}{m_2}-\frac{X_1}{m_1},\\
&\ddot{x}_3+\gamma y_3-(q^2+r^2)x_3+pqy_3+2r\dot{y}_3=\frac{X_3}{m_3}-\frac{X_1}{m_1},\\
&\ddot{y}_3-\gamma x_3-(r^2+p^2)y_3+qpx_3-2r\dot{x}_3=\frac{Y_3}{m_3}-\frac{Y_1}{m_1}.
\end{aligned}\right\}\quad(10a)$$

where:

$$\left.\begin{aligned}
&-\gamma x_2+qpx_2-2r\dot{x}_2=\frac{Y_2}{m_2}-\frac{Y_1}{m_1},\quad \beta x_2+rpx_2+2q\dot{x}_2=\frac{Z_2}{m_2}-\frac{Z_1}{m_1},\\
&-(\alpha y_3-\beta x_3)+r(px_3+qy_3)-2\,p\dot{y}_3-q\dot{x}_3)=\frac{Z_3}{m_3}-\frac{Z_1}{m_1},
\end{aligned}\right\}\quad(10b)$$

$$\left.\begin{aligned}
&p[\Sigma m\Sigma m(y^2+z^2)-(\Sigma my)^2-(\Sigma mz)^2]-q(\Sigma m\Sigma mxy-\Sigma mx\Sigma my)\\
&-r(\Sigma m\Sigma mzx-\Sigma mz\Sigma mx)=\Sigma m\Sigma m(y\dot{z}-x\dot{y})-(\Sigma my\Sigma m\dot{z}-\Sigma mz\Sigma m\dot{y}),\\
&\text{with two similar expressions for } y\text{- and } z\text{-axis.}
\end{aligned}\right\}\quad(10c)$$

This way of writing the equations is quite equivalent to the process of elimination described above. As before, α, β, γ, p, q and r must be considered as auxiliary quantities; they should be solved for from (10b) and (10c) and substituted in (10a), which is possible in view of the fact that (10b) are linear in α, β, γ, and (10c) in p, q, and r.

This leads to *3n−6 differential equations which are all of the second order* and contain only the 3n−6 coordinates and their derivatives. We were unable to obtain such a result in §2 where Föppl's hypothesis was not introduced.

Eqs. (10) are derived from the Newtonian Eqs. (2) and Föppl's conditions (7) and (8). All the mathematical processes of the derivation are reversible. This is not immediately obvious for the elimination of $\dot{p}$, $\dot{q}$ and $\dot{r}$ by putting $\dot{p}=\alpha$, $\dot{q}=\beta$, $\dot{r}=\gamma$. However it is proved in §8b, that the Eqs. (10) imply that

$$\alpha=\dot{p}\quad\beta=\dot{q}\quad\gamma=\dot{r},\qquad(11)$$

therefore the whole mathematical process is reversible. In other words, the Eqs. (10) are the analytical equivalent of Newton's equations (2) with Föppl's conditions (7) and (8).

The points 1, 2 and 3, by which the system of axes XYZ is fixed, are quite arbitrary. Therefore the Eqs. (10), transformed to a system of axes fixed by three other material points k, l and m, remain the same, only the indices 1, 2 and 3 should be replaced by k, l and m. This can be proved directly from Eqs. (10) by transformation.

5. *The equations in vector form. Application to a rigid body.* The first three Eqs. (10a) and Eqs. (10c) can be written in vector form. This cannot be done for the other equations, because of the asymmetry arising from the definition of the system of coordinates XYZ. If $\mathbf{r}_\nu$ represents the radius vector of the point ν, with components x_ν, y_ν, z_ν, and $\mathbf{F}_\nu$ the force on point ν, components X_ν, Y_ν, Z_ν, the *equations in vector form* become

$$\ddot{\mathbf{r}}_\nu - \mathbf{a}\times\mathbf{r}_\nu + \mathbf{w}\times(\mathbf{w}\times\mathbf{r}_\nu) - 2\mathbf{w}\times\dot{\mathbf{r}}_\nu = \frac{\mathbf{F}_\nu}{m_\nu} - \frac{\mathbf{F}_1}{m_1}, \tag{10a}$$

where $\mathbf{a}$, components α, β, γ, and $\mathbf{w}$, components p, q, r are given by (10b) and

$$\Sigma m \Sigma \mathbf{r}\times m(\mathbf{w}\times\mathbf{r}) - \Sigma m\mathbf{r}\times\Sigma m(\mathbf{w}\times\mathbf{r}) = \Sigma m\Sigma\mathbf{r}\times m\dot{\mathbf{r}} - \Sigma m\mathbf{r}\times\Sigma m\dot{\mathbf{r}}. \tag{10c}$$

The sign $\times$ designates a vector product.

There is no harm in writing down the vector Eqs. (10a) for all the n points, if the additional coordinates x_1, y_1, z_1, y_2, z_2, z_3, which are all zero, are introduced. But then the number of coordinates is no longer the minimum number $3n-6$.

The equation (11) in vector form becomes:

$$\mathbf{a} = \dot{\mathbf{w}} \tag{11}$$

In the special case in which the material points 1, 2 and 3 are three points of a *rigid body*, the equations of motion (10) can be reduced to

$$\ddot{\mathbf{r}}_\nu - \mathbf{a}\times\mathbf{r}_\nu + \mathbf{w}\times(\mathbf{w}\times\mathbf{r}_\nu) - 2\mathbf{w}\times\dot{\mathbf{r}}_\nu = \frac{\mathbf{F}_\nu}{m_\nu} - \frac{\mathbf{F}_1}{m_1}, \tag{12a}$$

where $\mathbf{a}$, components α, β, γ and $\mathbf{w}$, components p, q, r are given by

$$\left.\begin{aligned} -A\alpha + (C-B)qr &= L, \\ -B\beta + (A-C)rp &= M, \\ -C\gamma + (B-A)pq &= N. \end{aligned}\right\} \tag{12b}$$

$$\Sigma m \Sigma \mathbf{r}\times m(\mathbf{w}\times\mathbf{r}) - \Sigma m\mathbf{r}\times\Sigma m(\mathbf{w}\times\mathbf{r}) = \Sigma m\Sigma\mathbf{r}\times m\dot{\mathbf{r}} - \Sigma m\mathbf{r}\times\Sigma m\dot{\mathbf{r}}. \tag{12c}$$

This result can be obtained either by restricting the general equations (10) (See §8 c) or by replacing equation (4b) of §2 by Euler's dynamical equations for the angular velocity, components $-p$, $-q$, $-r$.

The index 1 at the right hand side refers to the material point at the origin of the system of coordinates. All the other indices ν refer to the material points in space outside the body of reference. The axes X, Y and Z are parallel to the principal axes of inertia; A, B and C represent the principal moments of inertia; L, M and N the components of the external moment.

6. *The system of relative mechanics founded on the equations of motion obtained. Ptolemy's system. Kepler's system. Newton's system with Föppl's conditions.* The general equations (10) in relative coordinates,

obtained in §5, can be *postulated as the fundamental equations for a system of mechanics*. They hold for the motion of any material point ν in space, e.g. for a "fixed" star as well as for a body on the earth. The three material points 1, 2 and 3, through which the system of axes XYZ is chosen, are quite arbitrary; this system may be laid through the sun and two stars, but it may be fixed to a ship, a train, an elevator, or an airplane performing acrobatic stunts, just as well. One might even refer the motion to a part of his own body, e.g. his head. This is perhaps the most direct system of reference in the sense that it corresponds most closely to the conception of bodies in space and their motion which we derive from our visual sensations.

Observations, however, are mostly made with instruments mounted in a laboratory or observatory, which is rigidly connected with the earth. For this reason we shall give some special attention to the case in which the system of reference XYZ is fixed to the earth, which leads to the conception known as *Ptolemy's system*.

Ptolemy's system. We assume:

(1) That the vectors $\mathbf{F}_\nu$, components X_ν, Y_ν, Z_ν, are the same as in the Newtonian theory.

(2) That the earth with sufficient accuracy may be considered as a rigid body.

(3) That there is a distance r_0 beyond which the relative rotation of the lines joining the observer with the stars becomes of lower order of magnitude compared with the motions we are dealing with.

The third assumption is an extension of an experimental fact to those cases in which the stars have not yet been observed, either on account of their being too far away, or because the amount of light emitted is too small. Stars at a distance r, larger than r_0, from the observer are indicated as "distant" stars, the remaining ones as "nearby."

Assumption 2, when applied to the centre of gravity G of the earth, gives from equation (10a)

$$\mathbf{F}_G/m_G-\mathbf{F}_1/m_1=-\dot{\mathbf{w}}\times\mathbf{r}_G+\mathbf{w}\times(\mathbf{w}\times\mathbf{r}_G), \tag{13}$$

where, in virtue of (11), $\mathbf{a}$ has been replaced by $\mathbf{w}$. Moreover, it justifies the use of Eq. (12) for a rigid body.

Eq. (12c), together with assumption 3, enables us to express $\mathbf{w}$ in terms of observed motions. The radius vectors of all distant stars in XYZ have the same angular velocity $\mathbf{u}$ and angular acceleration $\dot{\mathbf{u}}$, apart from deviations of higher order (Assumption 3). For stars nearby there is a deviation $\Delta\mathbf{u}$ from this value $\mathbf{u}$. The equation (12c) becomes therefore

$$\Sigma m\Sigma\mathbf{R}\times m(\mathbf{w}\times\mathbf{R})=\Sigma m\Sigma\mathbf{R}\times m(\mathbf{u}\times\mathbf{R})+\Sigma m\Sigma\mathbf{R}\times m(\Delta\mathbf{u}\times\mathbf{R}) \tag{14}$$

where $\mathbf{R}_\nu=\mathbf{r}_\nu-\Sigma m\mathbf{r}/\Sigma m$.

Observation shows that the number of these nearer stars having appreciable $\Delta\mathbf{u}$ is very small compared with the number of distant stars, and in general the $\mathbf{R}_\nu$ is of the same order. Neglecting for this reason the last term of the righthand side, Eq. (14) reduces to

$$\mathbf{u} = \mathbf{w}, \tag{15}$$

which means that, apart from deviations of a higher order, the auxiliary quantity $\mathbf{w}$ is equal to the angular velocity $\mathbf{u}$ of the system of distant stars.

Replacing (12c) by (15), the equation of motion (12) for the motion referred to a rigid body becomes in virtue of (13) and (11):

$$\ddot{\mathbf{r}}_\nu - \dot{\mathbf{u}}\times\mathbf{r}_\nu + \mathbf{u}\times(\mathbf{u}\times\mathbf{r}_\nu) - 2\mathbf{u}\times\dot{\mathbf{r}}_\nu = \mathbf{F}_\nu/m_\nu - \mathbf{F}_G/m_G - \dot{\mathbf{u}}\times\mathbf{r}_G + \mathbf{u}\times(\mathbf{u}\times\mathbf{r}_G), \tag{16a}$$

where

$$\left.\begin{aligned} -A\dot{p} + (C-B)qr &= L, \\ -B\dot{q} + (A-C)rp &= M, \\ -C\dot{r} + (B-A)pq &= N. \end{aligned}\right\} \tag{16b}$$

where p, q and r represent the components of $\mathbf{u}$; A, B and C, the principal moments of inertia of the earth, and L, M, and N, the components of the external moment of force on the earth.

Eqs. (16b) are Euler's dynamical equations, with the only difference that p, q and r have the signs changed. They give therefore, for the alteration of $\mathbf{u}$ (p, q, r), the usual precession and nutation of the Newtonian theory. The sign, however, is reversed, as we do not deal with motion of the earth, but with the motion of the distant stars with respect to the earth.

The remaining Eq. (16a) may be written

$$\ddot{\mathbf{r}}_\nu = \dot{\mathbf{u}}\times\mathbf{r}_\nu - \mathbf{u}\times(\mathbf{u}\times\mathbf{r}_\nu) + 2\mathbf{u}\times\dot{\mathbf{r}}_\nu + \frac{\mathbf{F}_\nu}{m_\nu} - \frac{\mathbf{F}_G}{m_G} - \dot{\mathbf{u}}\times\mathbf{r}_G + \mathbf{u}\times(\mathbf{u}\times\mathbf{r}_G). \tag{17}$$

The complex motion, given by this equation, may be interpreted as consisting of a number of simpler motions, superimposed kinematically. Table I gives this interpretation for different bodies ν, considered as material points.

The first three terms of the acceleration $\ddot{\mathbf{r}}_\nu$ can be made to disappear by giving to the new system of axes an angular velocity $-\mathbf{u}$, angular acceleration $-\dot{\mathbf{u}}$. They are accordingly interpreted as a rotation $\mathbf{u}$. The other terms are interpreted in a similar way. We notice that the rotation of the pendulum plane in Foucault's experiment and the daily motion of celestial bodies are represented by the same term. For planets we obtain, besides daily motion, precession and nutation, a kind of epicyclic motion consisting of two superimposed elliptical motions, with corrections for the influence of other planets. The second

ellipse of this epicyclic motion follows from the same term as the yearly parallax. Daily parallax for celestial bodies and centrifugal acceleration of a body on the earth are represented by the term d, which is numerically the same for all bodies. They are therefore identical motions. Forces not due to gravitation have a place in the system; they may work on bodies on the earth or on celestial bodies. No effects different from those of the Newtonian theory are obtained, as might be expected.

TABLE I.

PTOLEMY'S SYSTEM

Contribution of each of the components of the acceleration of a point ν.

Body	$a=$ $\dot{\mathbf{u}}\times\mathbf{r}_\nu$	$b=$ $-\mathbf{u}\times(\mathbf{u}\times\mathbf{r}_\nu)$ $+2u\times\dot{\mathbf{r}}$	$c=$ $\mathbf{F}_\nu/m_\nu-\mathbf{F}_G/m_G$	$d=$ $-\dot{\mathbf{u}}\times\mathbf{r}_G$ $+\mathbf{u}\times(u\times\mathbf{r}_G)$
star	precession and nutation	daily motion	proper motion and yearly parallax	daily parallax
sun	"	"	elliptical orbit, with corrections for influence of planets.	"
planets		"	first ellipse, with corrections for planets; second ellipse, same for all planets, with corr. — corrected epicyclic motion	"
moon	"	"	elliptic orbit, with corrections for influence of sun and planets	"
bodies on earth	"	rotation of pendulum plane in Foucault's exp., deviation of falling bodies, of projectiles, etc.	acceleration of gravity influence of sun and moon: tides. Influence of all other forces $\mathbf{F}_\nu$, on body ν, not due to gravity.	centrifugal acceleration (time of swing of pendulum)

Kepler's system. The sun is taken as origin (point 1) of the system XYZ and the direction of the axes is fixed by two distant stars 2 and 3. We make, as before, assumptions 1 and 3. In the system XYZ the radius vectors of all distant stars have zero rotation, apart from deviations of higher order. For nearer stars let this rotation be $\Delta\mathbf{u}$. From equation (10c) we obtain, then, in the same way as for *Ptolemy's* system the Eq. (15), which leads to

$$\mathbf{u}=\mathbf{w}=0.$$

Substituting this in the Eqs. (10a), we obtain, apart from deviations of a higher order

$$\ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu/m_\nu - \mathbf{F}_s/m_s, \tag{18}$$

where S stands for the center of the sun. This is the usual equation of motion for planets in Kepler's system.

Newton's system with Föppl's conditions. In this paragraph we take the point of view of the observer. He uses relative coordinates to describe the observed motion and for him, the equations of motion (10) in relative coordinates are therefore fundamental.

The observer may define a new system $X'Y'Z'$ in the following way. Let the radius vector of the origin of $X'Y'Z'$ be $\mathbf{r}_i$, its rotational velocity $\mathbf{w}_i$, both referred to the fundamental system XYZ. Then these quantities are given by:

$$\mathbf{r}_i = \Sigma m\mathbf{r}/\Sigma m, \tag{19a}$$

$$\Sigma m \Sigma \mathbf{r} \times m(\mathbf{w}_i \times \mathbf{r}) - \Sigma m\mathbf{r} \times \Sigma m(\mathbf{w}_i \times \mathbf{r}) = \Sigma m \Sigma \mathbf{r} \times m\ddot{\mathbf{r}} - \Sigma m\mathbf{r} \times \Sigma m\dot{\mathbf{r}}. \tag{19b}$$

The fundamental equations (10), referred to these new axes $X'Y'Z'$ become then:

$$m_\nu \ddot{\mathbf{r}}_\nu' = \mathbf{F}_\nu' \tag{20}$$

where moreover from (19a) and (19b) follow the conditions:

$$\Sigma m\mathbf{r}' = 0, \qquad \Sigma \mathbf{r}' \times m\dot{\mathbf{r}}' = 0. \tag{21}$$

This can be proved by performing the mathematical operations of §5 in the reverse order.

Eqs. (20) are the Newtonian equations (2); Eqs. (21) are Föppl's conditions (7) and (8).

In this way the observer has defined a system for which Newton's equations hold (inertial system) entirely in terms of relative coordinates, in which the observed motion is expressed. In a similar way Föppl[1] defined an inertial system on an entirely relative basis, though he did not have the equations in relative coordinates from which to start.

7. *Newton's equations of motion, transformed to an invariant form. Applications to special systems of axes.* When we take the standpoint that mechanics should deal with observed motions (§1), there are certain *difficulties* connected with the *definition of the system of axes*, if *Newton's* mechanics in its original form is used.

These difficulties may be removed in different ways:

(1) By defining the inertial systems in terms of observed motions, consequently on a relative basis. Such a solution by Föppl was mentioned in the preceding paragraph.

(2) By introducing the relative coordinates, in which the observed motion is expressed, in the equations of motion. A satisfactory solution was given by equations (10).

A third solution is obtained by writing down the *equations of motion in an invariant form.* Then the system of axes needs no specification at all.

The Newtonian equations of motion

$$m_\nu \ddot{\mathbf{r}}_\nu' = \mathbf{F}_\nu', \tag{20}$$

together with Föppl's conditions

$$\Sigma m\mathbf{r}' = 0, \qquad \Sigma \mathbf{r}' \times m\dot{\mathbf{r}}' = 0, \tag{21}$$

are transformed to an *arbitrary system of axes* XYZ. Then the following equations are obtained:

$$\ddot{\mathbf{r}}_\nu - \dot{\mathbf{w}} \times \mathbf{r}_\nu + \mathbf{w} \times (\mathbf{w} \times \mathbf{r}_\nu) - 2\mathbf{w} \times \dot{\mathbf{r}}_\nu - \ddot{\mathbf{s}}_\nu = \frac{\mathbf{F}_\nu}{m_\nu}, \tag{22a}$$

$$\mathbf{s} = \Sigma m\mathbf{r}/\Sigma m, \tag{22b}$$

$$\Sigma m \Sigma \mathbf{r} \times m(\mathbf{w} \times \mathbf{r}) - \Sigma m\mathbf{r} \times \Sigma m(\mathbf{w} \times \mathbf{r}) = \Sigma m \Sigma \mathbf{r} \times m\dot{\mathbf{r}} - \Sigma m\mathbf{r} \times \Sigma m\dot{\mathbf{r}}. \tag{22c}$$

In these equations $\mathbf{r}_\nu$ (x_ν, y_ν, z_ν) is the radius vector of point ν in the new system of axes, $\mathbf{s}$ the radius vector of the origin of $X'Y'Z'$ and $\mathbf{w}$ its angular velocity, both referred to XYZ. Regarding, however, $\mathbf{s}$ and $\mathbf{w}$ as auxiliary quantities, which are to be solved for from (22b) and (22c) and substituted in (22a), the $3n$ equations (22) contain the $3n$ coordinates of the material points only. They are therefore *equations of motion* and, as the system of axes XYZ is quite arbitrary, these equations are *invariant.*

Eqs. (22) hold for all systems of reference possible. For this reason it is convenient to use them as a starting point for a discussion of all special systems of axes.

A system of axes can be specified by restricting conditions. By introducing these conditions in (22) the equations of motion for that special system of axes are obtained. Of course it may be that the equations thus obtained can be simplified by purely analytical operations; the method of procedure does not ascertain the simplest form possible.

If our purpose in specifying the system of axes is to obtain a simple form of the equations of motion, a restriction which almost immediately suggests itself is

$$\Sigma m\mathbf{r} = 0, \qquad \Sigma \mathbf{r} \times m\dot{\mathbf{r}} = 0,$$

which is Föppl's definition of a *principal system of reference.* The equation reduces then to the Newtonian form $m_\nu \ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu$.

But we might wish to refer our motion to the *principal axes of inertia of the universe*; the restricting conditions are then:

$$\left.\begin{aligned} \Sigma mx &= 0, \quad \Sigma my = 0, \quad \Sigma mx = 0, \\ \Sigma myz &= 0, \quad \Sigma mzx = 0, \quad \Sigma mxy = 0, \end{aligned}\right\} \tag{23}$$

which, introduced in (22), lead to the equations of motion:

$$\left.\begin{aligned} &\mathbf{r}_\nu - \dot{\mathbf{w}}\times\mathbf{r}_\nu + \mathbf{w}\times(\mathbf{w}\times\mathbf{r}_\nu) - 2\bar{\mathbf{w}}\times\dot{\mathbf{r}}_\nu = \frac{\mathbf{F}_\nu}{m_\nu}, && (24a)\\ &\text{where the components } p, q, \text{ and } r \text{ of } \mathbf{w} \text{ are given by} \\ &p = \frac{M_x}{I_x}, \quad q = \frac{M_y}{I_y}, \quad r = \frac{M_z}{I_z}. && (24b) \end{aligned}\right\} \quad (24)$$

M_x, M_y, M_z represent the components of the angular momentum of the universe and I_x, I_y, I_z its principal moments of inertia.

Finally, the need of being as close to the observations as possible might induce us to introduce *relative coordinates* as those of §2. The restricting conditions are then:

$$x_1 = 0,\ y_1 = 0,\ z_1 = 0,\ y_2 = 0,\ z_2 = 0,\ z_3 = 0; \qquad (3)$$

and this leads in a rather simple way to the equations of motion (10) in relative coordinates.

The main use of invariant equations seems to be that they shorten the analytical procedure in cases such as the examples given here. Besides this, they offer a solution of the difficulties connected with the definition of the system of axes in ordinary Newtonian mechanics. This solution, however, can not be regarded as a relativation of motion such as is required by observation (see §1).

8. *Proof of some of the theorems used. Case of plane motion.* Of some statements made the proof will be given here. In general all equations used are in vector form (see §5).

a. *Order of the system of Eqs. (4) of §2.* There are $3n-6$ coordinates occurring in the $3n-6$ equations (4a) of the second order; moreover, we have the six auxiliary quantities, occurring in three ordinary equations and three differential equations of the first order (4b). The system consisting of (4a) and (4b) is therefore of order $2(3n-6)+3=6n-9$.

b. *Proof that equations (10) contain the conclusion*

$$\alpha = \dot{p},\quad \beta = \dot{q},\quad \gamma = \dot{r},\quad \text{or}\quad \mathbf{a} = \dot{\mathbf{w}}. \qquad (11)$$

Without going back to the original derivation of (10), Eq. (11) can be proved to be a direct consequence of Eqs. (10a) and (10c). One may convince himself of this by applying the operation $\Sigma m\, \Sigma_\nu\, \mathbf{r}_\nu \times m_\nu$ to equation (10a) written as $\bar{\varphi} = \mathbf{F}_\nu/m_\nu - \mathbf{F}_1/m_1$, and the operation $\dot{\bar{\gamma}} - \mathbf{w}\times\bar{\gamma}$ to equation (10c) written as $\boldsymbol{\gamma} = 0$. Two other equations are thus obtained. The right hand side of the first equation, in view of the law of action and reaction (§3), becomes equal to $-\Sigma m(\Sigma m\ \mathbf{r}\times)\ \mathbf{F}_1/m_1$, which by (10a) can be put in the form $(\Sigma m\ \mathbf{r}\times)\ \Sigma m\bar{\varphi}$. After applying the rule of vector algebra $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$, the only difference between the two equations is that the first one has $\mathbf{a}$, where the second

one has $\dot{\mathbf{w}}$. Therefore the $\mathbf{a}$, solved from equation 1, gives the same value as $\dot{\mathbf{w}}$ from equation 2. Therefore: $\mathbf{a}=\dot{\mathbf{w}}$, which is equation (11).

c. *Equation (12) obtained as a restriction of (10).* Take for system XYZ the three principal axes of inertia of the body of reference. For a point ϵ of this body the condition of rigidity gives $\dot{\mathbf{r}}_\epsilon=0$, $\ddot{\mathbf{r}}_\epsilon=0$. In virtue of this, equation (10), applied to all points ϵ, leads to

$$-\Sigma_\epsilon \mathbf{r}_\epsilon\times m_\epsilon(\mathbf{a}\times\mathbf{r}_\epsilon)+\Sigma_\epsilon\mathbf{r}_\epsilon\times m_\epsilon\{\mathbf{w}\times(\mathbf{w}\times\mathbf{r}^\epsilon)\}=\Sigma_\epsilon\mathbf{r}_\epsilon\times\mathbf{F}_\epsilon, \tag{25}$$

which written out for the axes X, Y and Z is equation (12b).

Eq. (10b) can be left out for this proof, as it is a consequence of (10a) and the condition of rigidity, expressed as $\mathbf{r}_\epsilon=$constant. The use of vectors however implies the introduction of the additional coordinates $x_1, y_1, z_1, y_2, z_2, z_3$, which are all zero (See §5).

d. *Case of plane motion.* In a former publication equations corresponding to (4) of §2 and (10) of §4 have been derived.[5] The origin of the system of axes XY is at point 1, its axis of X passes permanently through point 2. Then Eqs. (10) e.g. reduce to

$$\left.\begin{aligned} x_2(\ddot{x}_\nu-p^2x_\nu)-2p(\dot{x}y_{2\nu}-x_2\dot{y}_\nu)&=x_2\left(\frac{X_\nu}{m_\nu}-\frac{X_1}{m_1}\right)+y_\nu\left(\frac{Y_2}{m_2}-\frac{Y_1}{m_1}\right),\\ x_2(\ddot{y}_\nu-p^2y_\nu)+2p(\dot{x}_2x_\nu-\dot{x}_2x_\nu)&=x_2\left(\frac{Y_\nu}{m_\nu}-\frac{Y_1}{m_1}\right)-x_\nu\left(\frac{Y_2}{m_2}-\frac{Y_1}{m_1}\right),\end{aligned}\right\}\ (26a)$$

where:

$$p=\frac{\Sigma m\Sigma m(x\dot{y}-y\dot{x})-(\Sigma mx\Sigma m\dot{y}-\Sigma my\Sigma m\dot{x})}{\Sigma m\Sigma m(x^2+y^2)-(\Sigma mx)^2-(\Sigma my)^2}, \tag{26b}$$

(26)

which are the analytical equivalent of Newton's equations of motion with Föppl's hypothesis.

9. *Discussion of previous work done on this subject.* Most of the work done on this subject can be considered to consist of two parts:

(1) A negative part, being a criticism of Newton's idea of absolute motion. The main argument is an empirical one, running as follows. The observed motion is relative. Mechanics should deal with observed motions. Therefore mechanics should deal with relative motion in this sense.

(2) An attempt to define the inertial systems, to which motion in Newton's mechanics is referred, in terms of this observed motion, consequently on a relative basis.

[5] Zanstra, Proc. Kon. Acad. van Wet., Amsterdam **23**, 1421, 1921. The same article appeared in the Ann. der Physik (IV) **70**, 153, 1923. As a slight alteration in the formula was made, the reader is referred to the formulas (8) and (9) of the latter article, which are the formulas (4) and (10), derived here, specialized for the plane.

Of the work of some of the main authors we shall give here a very brief review.[6]

Newton's younger contemporary Berkeley[7] criticized Newton's idea of absolute motion. His main argument is the empirical one, that motion in all practical cases of observation is referred to bodies in space. As regards the positive part of his work, he refers to the notion of philosophers who think that ultimately the motion is referred to "the utmost unmoved walls or shell" of "the corporeal world," which in their conception, is finite. This seems to correspond to Copernicus' conception, part of which may be traced back to Aristotle.[8]

In more recent times, since 1870, the interest in these questions was revived by Neumann,[9] who calls attention to the fact that in Newton's mechanics the system of axes to which the motion is referred needs to be specified more clearly. He proposes to refer the motion to a hypothetical body "Alpha." At the same time he wants to retain the idea of absolute motion.

Maxwell,[10] though working out a relative conception for the translation, retains the idea of an absolute rotation, induced to this by experiments like Newton's pail experiment.

Lange and Mach have done much to spread a relative conception of motion. Lange[6] introduced the name "inertial system" and suggested a way of specifying inertial systems experimentally.[11] If we would have trial bodies on which no force was acting, the condition that three of these bodies should move in a straight line would be sufficient to fix inertial systems. In Lange's work much use is made of Newton's "Law of inertia," a principle justly criticised by Mach.

Mach[6] gave an extended criticism of the principles of Newton's mechanics. He reduces the number of fundamental theorems to a minimum and formulates them in a more satisfactory way. For this reference must be made to his original work. We only wish to mention here his interpretation of Newton's pail experiment, where he correlates the alteration in the level of the water with the motion of the pail with respect to the earth and the stars.

[6] For a more complete treatment see: L. Lange, Geschichtliche Entwickelung des Bewegungsbegriffs; E. Mach, Die Mechanik in ihrer Entwickelung; and Enzyklopedie der Math. Wiss. VI 1, p. 30.

[7] G. Berkeley, The Principles of Human Knowledge, Sections 111 et seq.

[8] L. Lange. loc. cit.[6]

[9] C. Neumann, Ueber die Prinzipien der Galilei-Newtonschen Mechanik.

[10] Maxwell, Matter and Motion.

[11] H. Seeliger, Sitz. Math. Phys. Kl. Bayr. Acad. Wiss. **36**, 85, 1906.

Boltzmann's[12] suggestion to refer the motion to the principal axes of inertia of the universe would be of great importance, if this system of reference would lead to the Newtonian equations. A form of the equations of motion in this system of reference, equations which are the analytical equivalent of Newton's equations with Föppl's hypothesis, is given in §7.

Föppl[13] in making his hypothesis that the total angular momentum of the universe vanishes permanently for a principal system of reference (§3), gave a satisfactory solution of the problem of how to specify inertial systems in terms of observed motions (see §6). In §3 it has been assumed that the number of material points in space is finite. It seems, however, that all the results obtained in this article can be maintained for an infinite number of points when in addition to that it is assumed that the rotation of the radius vector $\mathbf{r}_\nu$ of a point ν approaches a definite limit $\mathbf{u}$, when r_ν approaches infinity.

Föppl has pointed out that his hypothesis may lead to an experimental specification of the inertial system to which e.g. the motion of planets should be referred. This of course can only be done when an assumption is made concerning the bodies not yet observed. The inertial system, thus obtained, might be slightly different from the inertial system which is used now, but would presumably come very close to the empirical system.

The positive part of the work of all the authors mentioned above is a definition of inertial systems on a relative basis.

As regards the introduction of relative coordinates and the derivation of invariant equations attention should be called to Kolkmeyer's work.[14] The problems studied by him are more extensive in many respects; he deals e.g. with the elimination of absolute time.

Kolkmeyer works out the relativation of motion without introducing Föppl's hypothesis. For the general insight in the consequences of such a relativation the equations (4) of §2 may serve. As regards the invariant equations we may remark, that such equations can be obtained without introducing Föppl's hypothesis, but again they are of a higher order than equations (22). It does not seem that Kolkmeyer makes a sharp distinction between the introduction of relative coordinates, as distances, and the derivation of invariant equations.

[12] Boltzmann. Vorlesungen über die Prinzipien der Mechanik. Leipzig 1904. C. Neumann[9] made a similar suggestion, which however is quite contradictory to some of his other statements.

[13] A. Föppl, Vorlesungen über technische Mechanik, VI, Erster Abschnitt, Die relative Bewegung.

[14] N. H. Kolkmeyer, Eliminatie van de begrippen assenstelsel, lengte en tijd uit de vergelijkingen voor de planetenbewegingen, Thesis, Amsterdam, 1915.

The review given here shows that the present article is a continuation of a line of thought to which many authors have contributed. This line differs widely from the one followed in Einstein's theories, its primary need being to give a conception which is as close as possible to the way in which observations are actually performed. In the opinion of the author invariant equations do not offer a complete solution of the problem of relativation so as to satisfy the physicist. Such a solution is rather obtained by the introduction of relative coordinates in the equation of motion. This can be done in Newton's theory and there is reason to suppose that it can be done in Einstein's theories also, though the problem seems to be one of no small mathematical difficulty. The purpose of the present article is to show how it can be done in the rather simple case of Newtonian mechanics and for those who are interested in problems of this kind, the next thing is to treat electromagnetic and opticalphenomena by a similar method. In our opinion the expression of the motion of particles with respect to other particles is the result of a method of treatment, not the privilege of any special theory. We do not believe that such a method has been applied to Einstein's theories as yet. Let us take e.g. in Einstein's general theory the case where a heavy and a light particle are the only two bodies in space. The theory in its present form gives for the exact solution of this problem,[15] that the measured distance ρ of the particles and a certain angle φ are both certain definite functions of a time coordinate t, if the observer is at rest with respect to the heavy particle, and the question arises as to what the angle φ means, since the only measurable space quantity is the distance ρ. One may be able to evade this difficulty by assuming that the boundary conditions are satisfied if other bodies are present only at a large distance (practically infinity). It seems that instead of this an hypothesis similar to that of Föppl may be used to simplify the problem so as to give a definite answer. We are so bold as to suggest that as a result of such a general hypothesis one will obtain the present solution with the restriction $d\varphi/dt=0$.

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UNIVERSITY OF MINNESOTA,
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[15] K. Schwarzschild, Sitz. Preus. Akad. der Wiss., Berlin 1916, p. 189.